Received: *November 22, 2001,* Accepted: *November 27, 2001*

# Derivative expansion and renormalisation group flows

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Abstract: We study the convergence of the derivative expansion for flow equations. The convergence strongly depends on the choice for the infrared regularisation. Based on the structure of the flow, we explain why optimised regulators lead to better physical predictions. This is applied to $O(N)$-symmetric real scalar field theories in $3d$, where critical exponents are computed for all $N$. In comparison to the sharp cut-off regulator, an optimised flow improves the leading order result up to 10%. An analogous reasoning is employed for a proper time renormalisation group. We compare our results with those obtained by other methods.

Keywords: Renormalization Regularization and Renormalons, Field Theories in Lower Dimensions, Nonperturbative Effects.

JHEP11(2001)059

# Contents



JHEP11(2001)059

# 1. Introduction

The recent years have witnessed an important progress in the development of non-perturbative methods that provide a reliable description of systems with large effective couplings or divergent correlation lengths, problems which are difficult to handle, if at all, within standard perturbation theory. A particularly efficient method is known as the exact renormalisation group (ERG) [1]–[4] (for recent reviews, see [5, 6] for scalar and [7] for gauge theories), which grew out of the idea of coarse-graining quantum fields. In its modern form, the ERG flow for an effective action $\Gamma_k$ for bosonic fields is given by the simple one-loop expression

$$\partial_t \Gamma_k = \frac{1}{2} \mathrm{Tr} \left( \Gamma_k^{(2)} + R_k \right)^{-1} \partial_t R_k \, . \tag{1.1}$$

Here, $\Gamma_k^{(2)}$ denotes the second functional derivative of the effective action, $t = \ln k$ is the logarithmic scale parameter, and $R_k(q^2)$ is an infrared (IR) regulator at the momentum scale $k$. The regulator fulfils a few constraints (given below) to ensure that (1.1) interpolates between a microscopic initial action $\Gamma_\Lambda$ at the ultraviolet (UV) scale $k = \Lambda$, and the full quantum effective action $\Gamma$ in the infrared limit $k = 0$.

An application of the ERG formalism requires some approximations. A commonly used systematic approximation scheme is the derivative expansion of effective actions [8]. However, little is known about its convergence, because there is *a priori* no small parameter associated to it. Indeed, for the ERG flow (1.1), the derivative expansion implies an expansion of $\Gamma_k^{(2)}$ inside the momentum trace in powers of $q/k$, where $q$ is the loop momenta and $k$ the infra-red scale. The validity of such a procedure, and, consequently, the convergence of the derivative expansion, are *unavoidably* entangled with the particular choice for the IR regulator, since the regulator ensures that the momentum trace is peaked for momenta $q^2 \approx k^2$. For the computation of $\beta$-functions, this has been studied in [9]. More generally, and similar to perturbative QCD or truncations of Schwinger Dyson equations, approximate solutions of (1.1) depend spuriously on the IR regularisation [10]–[16]. A deeper understanding of the scheme dependence, and its link to the stability and convergence of the ERG flow, has been established recently [14, 15, 16]. These results are at the basis for reliable physical predictions based on the ERG formalism.

In this Letter, we study the convergence of the derivative expansion for $3d$ $\mathrm{O}(N)$-symmetric scalar theories. We compute universal critical exponents for all $N$ and different regularisations. Based on conceptual arguments, we explain why specific regulators are expected to provide better physical predictions already to leading order in the derivative expansion. A deeper link between the convergence and the optimisation of ERG flows is established. While most of our considerations are based on ERG flows, we also discuss the convergence for specific proper time flows. A detailed comparison with results from other methods and higher orders in the derivative expansion is also given.

## 2. Derivative expansion

In the context of flow equations, the derivative expansion is based on the assumption that higher derivative operators lead only to small corrections compared to lower order ones, which does not imply that the UV and IR degrees of freedom are the same. Hence, the anomalous dimension $\eta$ of the quantum fields should be small. Some physical systems are known where these assumptions are realised. Consider the universality class of $\mathrm{O}(N)$-symmetric real scalar theories in $d = 3$ dimensions, where $N = 0$ describes the physics of entangled polymers; $N = 1$ the Ising model (water); $N = 2$ the XY model ($\mathrm{He}^4$); $N = 3$ the Heisenberg model (ferromagnets). The case $N = 4$ is expected to describe the phase transition of QCD with two massless quark flavours.

At the Wilson-Fisher fixed point the critical properties of $\mathrm{O}(N)$-symmetric real scalar theories are characterised by universal critical exponents: $\nu_{\rm phys}$, given by the inverse of the negative eigenvalue of the stability matrix at criticality, and $\eta_{\rm phys}$, the anomalous dimension. It is known from experiment that $\eta_{\rm phys}$ is at most of the order of a few percent. Hence, it is believed that the derivative expansion is a good approximation for a reliable computation of universal critical exponents. Within the derivative expansion, the physical critical exponents at the scaling solution are



JHEP11(2001)059

computed as the series

$$\nu_{\rm phys} = \nu_0(\text{RS}) + \nu_1(\text{RS}) + \nu_2(\text{RS}) + \cdots \tag{2.1}$$

$$\eta_{\rm phys} = 0 + \eta_1(\text{RS}) + \eta_2(\text{RS}) + \cdots . \tag{2.2}$$

Here, the index corresponds to the order of the derivative expansion. To leading order, $\eta_0(\text{RS}) \equiv 0$. Notice that every single order in the expansion — due to the approximations employed — depends on the regulator scheme. The independence of physical observables on the regulator scheme (RS) can only be guaranteed in the limit where *all* operators of the effective action are retained during the flow. In turn, the physical values $\nu_{\rm phys}$ and $\eta_{\rm phys}$ are independent of the precise form of the infrared regulator. Hence, the infinite sum on the right-hand side adds up in a way such that the physical values are scheme independent. The convergence of the expansion (2.1) is best if a regulator is found such that the main physical information is contained in a few leading order terms.

For specific values of $N$, the derivative expansion becomes exact (and hence independent on the regulator). In the large-$N$ limit, the universal critical exponents are

$$\nu_{\rm phys} = 1 + \mathcal{O}(1/N) \tag{2.3}$$

$$\omega_{\rm phys} = 1 + \mathcal{O}(1/N) \tag{2.4}$$

$$\eta_{\rm phys} = 0 + \mathcal{O}(1/N) . \tag{2.5}$$

All subleading universal eigenvalues of the stability matrix at criticality are given as $\omega_n = 2n - 1 + \mathcal{O}(1/N)$ for $n \geq 1$ ($\omega \equiv \omega_1$), and all $\mathcal{O}(1/N)$ coefficients are RS dependent. Stated differently, the large-$N$ limit is so strong that all scheme dependence of universal quantities is suppressed as $\mathcal{O}(1/N)$. For the case $N = -2$, it is known that [17, 18]

$$\begin{aligned} \nu_{\rm phys} &= \frac{1}{2} + \mathcal{O}(N+2) \\ \eta_{\rm phys} &= 0 + \mathcal{O}(N+2) , \end{aligned} \tag{2.6}$$

for all regulators. All RS dependent corrections are supressed. In contrast to the large-$N$ limit, we find that the subleading eigenvalues $\lambda_n$ are not universal (see figure 2 below).

## 3. Renormalisation group flows

Next we use (1.1) to compute the leading order of (2.1) for all $N$. The flow trajectory of (1.1) in the space of all action functionals depends on the IR regulator function $R_k$. $R_k$ obeys a few restrictions, which ensure that the flow equation is well-defined,



JHEP11(2001)059

thereby interpolating between an initial action in the UV and the full quantum effective action in the IR. We require that

$$\lim_{q^2/k^2\to 0} R_k(q^2) \;>\; 0\,, \tag{3.1}$$

$$\lim_{k^2/q^2\to 0} R_k(q^2) \;\to 0\,, \tag{3.2}$$

$$\lim_{k\to\Lambda} R_k(q^2) \;\to \infty\,. \tag{3.3}$$

Equation (3.1) ensures that the effective propagator at vanishing field remains finite in the infrared limit $q^2 \to 0$, and no infrared divergences are encountered in the presence of massless modes. Equation (3.2) guarantees that the regulator function is removed in the physical limit, where $\Gamma \equiv \lim_{k\to 0}\Gamma_k$. Equation (3.3) ensures that $\Gamma_k$ approaches the microscopic action $S = \lim_{k\to\Lambda}\Gamma_k$ in the UV limit $k \to \Lambda$. We put $\Lambda = \infty$, and introduce for later convenience the dimensionless regulator function $r(y)$ as

$$R_k = Z_k \cdot q^2\, r(q^2/k^2)\,. \tag{3.4}$$

Here, we have also introduced a wave function renormalisation factor $Z_k$ into the regulator. We set $Z \equiv 1$ to leading order in the derivative expansion. A few explicit regulators are

$$r_{\rm opt}(y) = \left(\frac{1}{y} - 1\right)\theta(1-y) \tag{3.5}$$

$$r_{\rm power}(y|b) = y^{-b} \tag{3.6}$$

$$r_{\rm sharp}(y) = 1/\theta(1-y) - 1 \tag{3.7}$$

$$r_{\rm exp}(y|b) = 1/(\exp cy^b - 1) \tag{3.8}$$

$$r_{\rm mix}(y|b) = \exp[-b(y^{1/2} - y^{-1/2})]\,, \tag{3.9}$$

with $b \geq 1$ (and $c = \ln 2$). The optimised regulator (3.5) has been introduced in ref. [15]. It leads to better stability and convergence properties of ERG flows [14, 16]. The class of power-like regulators (3.6) for $b \geq 1$ is often used for analytical and numerical considerations [19]. However, for large momenta, the regulator decays only as a power law, as does $\partial_t R_{\rm power}$. This may lead to an insufficiency in the integrating out of momentum variables. Eq. (3.7) describes the sharp cut-off regulator. It corresponds to the limit $b \to \infty$ of both (3.6) and (3.8) [20]. The exponential regulator (3.8), which is also expected to have good convergence properties due to the exponential suppression of large momenta, is often used for numerical studies (e.g. [6, 21]). Finally, the class of mixed exponential regulator $r_{\rm mix}$ has been employed in [13].

With the main definitions at hand, we turn to the ERG flow for O($N$)-symmetric scalar theories to leading order in the derivative expansion — the local potential



JHEP11(2001)059

approximation. We are lead to the Ansatz

$$\Gamma_k = \int d^d x \left( U_k(\bar\rho) + \frac{1}{2} Z_k(\bar\rho)\partial_\mu\phi^a\partial_\mu\phi_a + \frac{1}{4} Y_k(\bar\rho)\partial_\mu\bar\rho\partial_\mu\bar\rho + \mathcal{O}(\partial^4)\right) \tag{3.10}$$

for the effective action $\Gamma_k$, with $\bar\rho = \frac{1}{2}\phi^a\phi_a$. For $N \neq 1$, there are two independent wave function factors $Z_k$ and $Y_k$ beyond the leading order in this expansion. To leading order in the derivative expansion, the flow equation (1.1) reduces to a flow $\partial_t U_k$ for the effective potential. Inserting (3.10) into (1.1), setting $Z \equiv Y \equiv 1$, and rewriting the flow for the effective potential in dimensionless variables $u = U_k/k^d$ and $\rho = \bar\rho/k^{d-2}$, we find

$$\partial_t u + du - (d-2)\rho u' = 2v_d(N-1)\ell(u') + 2v_d\ell(2\rho u'') \tag{3.11}$$

and $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$. Here, we have also performed the angle integration of the momentum trace. The functions $\ell(\omega)$ are given by

$$\ell(\omega) = \frac{1}{2}\int_0^\infty dy y^{d/2} \frac{\partial_t r(y)}{y(1+r)+\omega}\,, \tag{3.12}$$

where the integration over $y \equiv q^2/k^2$ corresponds to the remaining integral over the size of the loop momenta, and $\partial_t r(y) = -2yr'(y)$. All non-trivial information regarding the renormalisation flow and the RS are encoded in (3.12). In turn, all terms on the left-hand side of (3.11) do not depend on the RS. They simply display the intrinsic scaling of the variables which we have chosen for our parametrisation of the flow. For the regulators $r_{\rm opt}(y)$, $r_{\rm quart}(y) \equiv r_{\rm power}(y|2)$ and $r_{\rm sharp}(y)$, the flows (3.12) can be computed analytically. We find

$$\ell_{\rm opt}(\omega) = \frac{2}{d}(1+\omega)^{-1} \tag{3.13}$$

$$\ell_{\rm quart}(\omega) = \pi\,(2+\omega)^{-1/2} \tag{3.14}$$

$$\ell_{\rm sharp}(\omega) = -\ln(1+\omega)\,. \tag{3.15}$$

While (3.13) and (3.15) hold for any dimension, (3.14) holds only for $d = 3$. Notice that the functions $\ell(\omega)$ differ mainly in their asymptotic decay for large arguments.

## 4. Convergence

Now we address the convergence of (2.1), which is expected to be best if $|\nu_0(\mathrm{RS})| \gg |\nu_1(\mathrm{RS})| \gg \cdots$, as this would imply that the main physical information is contained within a few leading order terms of the expansion. Hence, it is mandatory, first, to understand the range within which $\nu_0(\mathrm{RS})$ varies as a function of the scheme [22], and, second, to discriminate those regulators for which $|\nu_1(\mathrm{RS})/\nu_0(\mathrm{RS})|$ is smallest. In refs. [14, 15, 16], we have argued that specific such regulators — based

JHEP11(2001)059

on an optimisation of the flow — are indeed available. Consider the flow (3.11), parametrised in terms of the scheme dependent functions (3.12), in the region of small (field) amplitudes. With small amplitudes, we denote those regions in field space where $|\omega| \lesssim \mathcal{O}(1)$, while $\omega \equiv u'$ or $u' + 2\rho u''$ are the amplitudes entering the functions (3.12). For example, a polynomial approximation of the scaling potential about the local minimum is viable for a determination of universal critical exponents [23], with the exception of flows based on problematic regulators like the sharp cut off [24, 14]. Note that the expansion is performed only for the presentation of our line of reasoning. It is certainly not needed for solving the equations. Expanding the flow (3.12) in powers of its argument, we find

$$\ell(\omega) = \sum_{n=0}^{\infty} a_n (-\omega)^n \tag{4.1}$$

with expansion coefficients ($y \equiv q^2/k^2$)

$$a_n = \int_0^\infty dy \frac{-y^{1+d/2}\, r'(y)}{[y(1+r)]^{n+1}}\,, \tag{4.2}$$

and all $a_n \geq 0$. For small and fixed $\omega$, only a few coefficients $a_n$ are required for a reliable approximation of the flow. Higher order corrections are smallest if $a_n/a_{n+1}$ is smallest. In the limit $n \to \infty$, this ratio becomes the radius of convergence of amplitude expansions,

$$C = \lim_{n\to\infty} (a_n/a_{n+1})\,. \tag{4.3}$$

In ref. [14], it has been shown that the limit (4.3) is given by

$$C = \min_{y \geq 0}\; y(1 + r(y))\,. \tag{4.4}$$

The result is easily understood: the function $y(1 + r(y))$, the regularised dimensionless inverse propagator at vanishing field, displays a "gap" due to the regularisation. Consequently, in the limit $n \to \infty$, the integrand in (4.2) is suppressed the least at the minimum of $y(1 + r)$, whence (4.4). Let us normalise all regulators by the requirement $r(1/2) = 1$, in order to compare their respective radii of convergence. (The normalisation differs from the one used in ref. [14].) From (4.4) and the normalisation condition, we obtain $C_{\text{OPT}} \equiv \max_{(\text{RS})} C = 1$. The gap is bounded from above. We refer to regulators with $C_{\text{OPT}} = 1$ as optimised. The extremisation of (4.4) is closely linked to a minimum sensitivity condition [16].

In ref. [14], we have computed the radii of convergence for different classes of regulators. For (3.5), we found $C_{\text{opt}} = 1$. For the sharp cut off, the result is $C_{\text{sharp}}/C_{\text{opt}} = 1/2 < 1$. In turn, for the quartic regulator, we have $C_{\text{quart}}/C_{\text{opt}} = 1$. Therefore, we expect better convergence properties for flows based on $r_{\text{quart}}$ and $r_{\text{opt}}$ than those based on $r_{\text{sharp}}$. A last comment concerns all regulators with $C = 1$.



JHEP11(2001)059

While their asymptotic limits (4.3) are the same, their approach to (4.3) still depends on the RS. The specific regulator (3.5) is distinguished because the limit (4.3) is attained for *all* values of $n$, and not only asymptotically. Therefore, we expect better results for flows based on $r_{\rm opt}$ in (3.5).

## 5. Anomalous dimension

Next, we consider higher order corrections to (2.1) implied through the anomalous dimension $\eta = -\partial_t \ln Z$. Here, $Z^{1/2}$ stands for the wave function renormalisation of the field. These corrections appear at second order in the derivative expansion. Once $\eta \neq 0$, the renormalised dimensionless field variable is $\rho = \frac{1}{2} Z_k \cdot \phi^a \phi_a / k^{d-2}$. Consequently, the term $(d-2)\rho u'$ on the left-hand side of (3.11) changes into $(d-2-\eta)\rho u'$. We also introduce $Z$ into the regulator, as done in (3.4). Therefore, the scale derivative $\partial_t r$ in (3.12) depends on $\eta$ as $\partial_t r = -2yr' - \eta\, r$. The new function $\ell(\omega, \eta)$ is Taylor expanded as

$$\ell(\omega,\eta) = \sum_{n=0}^{\infty} (a_n - \eta\, b_n)\, (-\omega)^n\,, \tag{5.1}$$

where $a_n$ are given by (4.2), and the higher order expansion coefficients $b_n \geq 0$ are

$$b_n = \frac{1}{2}\int_0^\infty dy \frac{y^{d/2}\, r(y)}{[y(1+r)]^{n+1}}\,. \tag{5.2}$$

The back coupling of higher order operators — parametrised by $\eta$ — is proportional to the ratio $b_n/a_n$. Consider this ratio in the limit of large $n$. We find

$$\lim_{n\to\infty} \left(\frac{b_n}{a_n}\right) = \left(\frac{-r(y)}{2yr'(y)}\right)_{y=y_{\rm min}} = \frac{1}{4}\,. \tag{5.3}$$

The first equality sign holds if the global minimum of the function $y(1+r)$, attained at $y_{\rm min}$, is *local* in momentum space, that is, not extending over an entire region around $y_{\rm min}$. The second equality sign holds, if, in addition, the regulator maximises (4.3). To confirm (5.3), it is useful to employ $0 = 1 + r + yr'$, which holds for optimised regulators at $y = y_{\rm min}$. We conclude that higher order corrections due to $\eta$ are suppressed for optimised regulators by an additional factor of (at least) $1/4$.

Let us consider two explicit examples. For the quartic regulator and any $d$, we find

$$\ell_{\rm quart}(\omega,\eta) = \left(1 - \frac{\eta}{4}\right) \ell_{\rm quart}(\omega)\,. \tag{5.4}$$

We conclude from (5.4) that $b_n/a_n = 1/4$ for *all* $n$. For the optimised regulator (3.5), the minimum of the function $y(1 + r_{\rm opt}(y))$ extends over the entire interval $y \in [0, 1]$.



JHEP11(2001)059

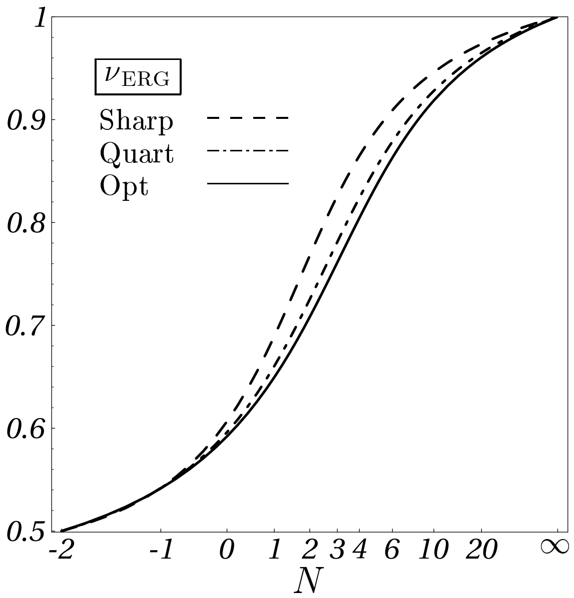


**Figure 1:** The critical exponent $\nu_{\rm ERG}$ for various regulators. The $N$-axis has been squeezed as $N \to (N+2)/(N+6)$ for display purposes.

Therefore, the reasoning which has lead to (5.3) is not applicable for (3.5). An explicit computation shows that [15]

$$\ell_{\rm opt}(\omega, \eta) = \left(1 - \frac{\eta}{d+2}\right) \ell_{\rm opt}(\omega)\,. \tag{5.5}$$

We emphasize that the back-coupling of the anomalous dimension is *dimensionally* suppressed in (5.5) by a factor $1/(d+2)$, or $1/5$ in $d = 3$ dimensions. Actually, $b_n/a_n = 1/(d+2)$ for *all* $n$. Hence, of all optimised regulators, the regulator $r_{\rm opt}$ is singled out due to the *additional* suppression of $\eta$ corrections by a factor $4/(d+2)$ in comparison to generic optimised flows. The back coupling is also consistently smaller than for the quartic regulator. Hence, we expect for optimised regulators that $|\nu_1({\rm RS})/\nu_0({\rm RS})|$ is smallest for $r_{\rm opt}$ in (3.5).

## 6. Critical exponents

Starting with (3.11), we have computed the critical exponents $\nu$ and $\omega$ at the Wilson-Fisher (WF) fixed point. For the numerics, we found it more convenient to use the flow for $u'$ instead of (3.11). In $d = 3$ dimensions and for arbitrary $N$, the flow equation (3.11) has two fixed points $u'_\star$ with $\partial_t u'_\star = 0$: the trivial or gaußian one, $u'_\star \equiv 0$, and the non-trivial WF fixed point with $u'_\star \neq 0$. Small perturbations about the WF fixed point have a discrete spectrum of eigenvalues. The negative eigenvalue corresponds to the unstable direction, and its negative inverse is given by the critical



JHEP11(2001)059

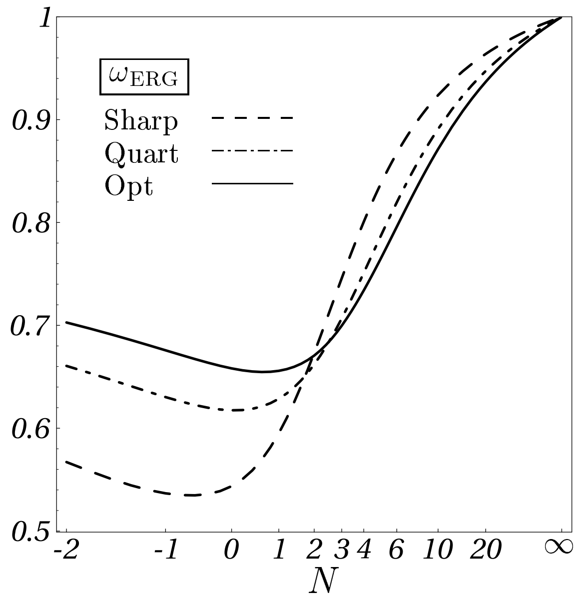


**Figure 2:** The smallest irrelevant eigenvalue at criticality, $\omega_{\rm ERG}$. The $N$-axis has been squeezed as $N \to (N+2)/(N+6)$ for display purposes.

exponent $\nu$. The exponent $\omega$ denotes the smallest irrelevant (and hence positive) eigenvalue. To leading order in the derivative expansion, the anomalous dimension is $\eta \equiv 0$.

Figure 1 shows our results for the exponent $\nu(N)$ for $-2 \leq N \leq \infty$, and for the three classes of regulators $r_{\rm opt}$, $r_{\rm quart}$ and $r_{\rm sharp}$. For the sharp cut off, our results agree with the findings of [25, 26]. For all regulators, the critical exponent $\nu(N)$ is a monotonically increasing function with $N$. For fixed $N$, the critical exponent $\nu$ depends on the RS. Notice that the curves of $r_{\rm opt}$ and $r_{\rm quart}$ are essentially on top of each other for $N \leq -1$. The results of ref. [13] are also worth mentioning. There, the authors computed the critical exponent $\nu$ for $N = 1$ and the classes of regulators (3.6), (3.8) and (3.9) for all $b \geq 1$, finding $\nu_{\rm sharp} \geq \nu_b > \nu_{\rm opt}$. More generally, the critical exponents are bounded from below $\nu_{\rm ERG} \geq \nu_{\rm opt}$ to leading order in the derivative expansion [22], and for $N \gtrsim -1$, the smallest value is attained through $r_{\rm opt}$. For $N \lesssim -1$, we note that the line for $r_{\rm sharp}$ is below those for $r_{\rm opt}$ and $r_{\rm quart}$. In figure 1, however, this small effect is barely visible.

Figure 2 shows the first irrelevant critical exponent $\omega(N)$ for $-2 \leq N \leq \infty$. In contrast to $\nu$, it is no longer a monotonic function of $N$. As a function of the regularisation, the turning point $\partial\omega/\partial N = 0$ moves from $N \approx 1$ ($r_{\rm opt}$) via $N \approx 0$ ($r_{\rm quart}$) to $N \approx -1$ ($r_{\rm sharp}$). We emphasize that all subleading eigenvalues for different RS join in the limit $N = \infty$, in consistency with the known result (2.4). However, this holds not true for $N = -2$, where $\omega$ found to be non-universal.

As a good measure for the scheme dependence, at any fixed $N$, we consider the range over which $\nu$(RS) and $\omega$(RS) vary as functions of the RS. Such a comparison is

JHEP11(2001)059

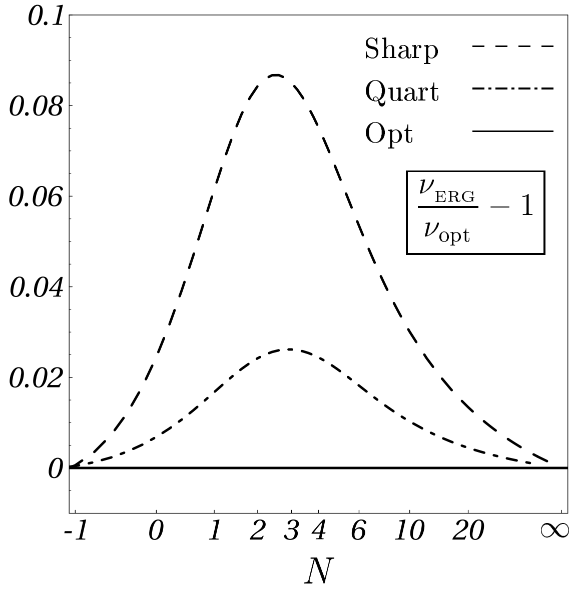


**Figure 3:** The relative improvement of the critical exponents $\nu_{\rm ERG}$ for various regulators in comparison with $\nu_{\rm opt}$.

sensible, because a variety of qualitatively different regulators are covered within the boundaries set by the sharp and the smooth optimised cutoff. In figure 3, we have displayed the ratio $\nu_{\rm ERG}/\nu_{\rm opt} - 1$ for the sharp cut off (dashed line) and the quartic (dashed-dotted line) regulator. For $\nu$, the boundaries are set by regulators with the largest (smallest) gap $C$, and hence by $r_{\rm sharp}$ and $r_{\rm opt}$ with $C_{\rm sharp}/C_{\rm opt} = \frac{1}{2}$. From figure 3, we conclude that the scheme dependence vanishes for $N = \infty$ and $N = -2$, consistent with the known results (2.3) and (2.6). In turn, the scheme dependence is largest around $N_{\rm max} \approx 2{-}3$. This suggests that a perturbative expansion around $N = \infty$ ($N = -2$) may be feasible for $N > N_{\rm max}$ ($N < N_{\rm max}$). The scheme dependence of $\omega$ (figure 2) turns out to be largest at $N = -2$, while it vanishes at $N = \infty$.

Based only on figure 3, it is not possible to decide which regulator would lead to a good estimate of the physical value within the given approximation. At this point, we take advantage of the reasoning presented in sections 4 and 5, where it has been argued that an optimised regulator leads to better convergence and stability properties of the flow, and to smaller higher order corrections. Therefore, $r_{\rm opt}$ provides a preferred choice. In this light, the comparison in figure 3 gives a quantitative idea on the improvement implied by an optimised choice for the regulator. With respect to the sharp cut off, the optimised regulator leads to a decrease of $\nu_{\rm ERG}$ up to 10% (or $3 - 10\%$ for $N \in [0, 4]$; the maximum is attained around $N \approx 2 - 3$). In comparison to the quartic regulator, the optimised regulator $r_{\rm opt}$ leads to a decrease of up to 3%, with the maximum around $N \approx 3$. For reasons given earlier, the relative decrease is smaller compared to the sharp cut off. The values for $\nu_{\rm opt}$ are indeed closest to the physical values, to leading order in the derivative expansion.



JHEP11(2001)059

## 7. Comparison

Results by other renormalisation group methods, and experiment, have been discussed in the literature. We compare the findings for $\nu_{\text{ERG}}$ and $\omega_{\text{ERG}}$ with ERGs for other regulators [25, 19, 26, 27, 13, 21], with the Polchinski RG [1, 10, 28], with a proper time RG [29, 30], all to leading and subleading order in the derivative expansion, and with experimental data, data from Monte Carlo simulations [32] and results from other field theoretical methods [33]. In figure 4 (figure 5), the critical index $\nu$ (subleading eigenvalue $\omega$) is displayed as predicted by various methods. For $\nu$, the range of data points is taken from [32]. For $N = 0, 3$ and 4, the boundary values stem from field theoretical predictions or MC simulations. For $N = 1$ and 2, the boundary values stem from experimental data. The bounds on field theoretical methods, reviewed in [33], are much tighter. They agree surprisingly well with the findings of [29] (within less than 1% for $\nu$ and around 5% for $\omega$, for all $N = 0, \cdots, 4$), and are represented by the short-dashed line in figures 4 and 5.

To order $\mathcal{O}(\partial^0)$, results for $\nu$ (figure 4) and $\omega$ (figure 5) are given for the sharp (medium dashed line), the quartic (dashed-dotted line) and the optimised (full line) regulator. In [16], it has been shown that the critical exponents from the Polchinski RG, at the present order, are equivalent to those from the optimised ERG. All ERG results for $\nu_{\text{ERG}}$ are systematically too large. Changing the ERG regulator from sharp via quartic to the optimised one, we notice in figure 4 that the curves bend down towards the values favoured by experiments and other methods. In particular, the results for (3.5) are closest to the physical values. This behaviour is fully consistent with the picture derived above: we expect that higher order corrections for $r_{\text{opt}}$ are smaller than those for $r_{\text{sharp}}$ and $r_{\text{quartic}}$. In figure 5, we notice that the $\omega_{\text{ERG}}$ curves for the sharp, the quartic and the optimised regulator bend towards the values preferred by other methods. The $N$ dependence of $\omega_{\text{ERG}}$ matches the $N$ dependence of the data only for small $N$. Still, the numerical agreement is acceptable, bearing in mind that $\omega_{\text{ERG}}$ is a subleading exponent like $\eta$.

To order $\mathcal{O}(\partial^2)$, the exponential regulator [21] leads to good predictions for $\nu$, for all $N$ considered (figure 4). No results for $\omega$ have been reported. Within the Polchinski RG, results for $N = 1$ have been given in [10, 28]. To order $\mathcal{O}(\partial^2)$, the results depend on two scheme-dependent parameters, which are determined by matching the anomalous dimension with the known experimental value, and by a minimum sensitivity condition. Then, the results for $\nu$ and $\omega$ agree well with other methods.

For the quartic regulator [19, 27], the results for $\nu$ agree less well with the available data, except for $N = 1$. For $N = 4$, the $\mathcal{O}(\partial^2)$ result for the quartic regulator is already worse than the $\mathcal{O}(\partial^0)$ result based on (3.5). The results for $\omega$ (figure 5) are even more sensitive and show a strong $N$ dependence. The findings of ref. [27] suggest that the derivative expansion in $d = 3$ dimensions with the

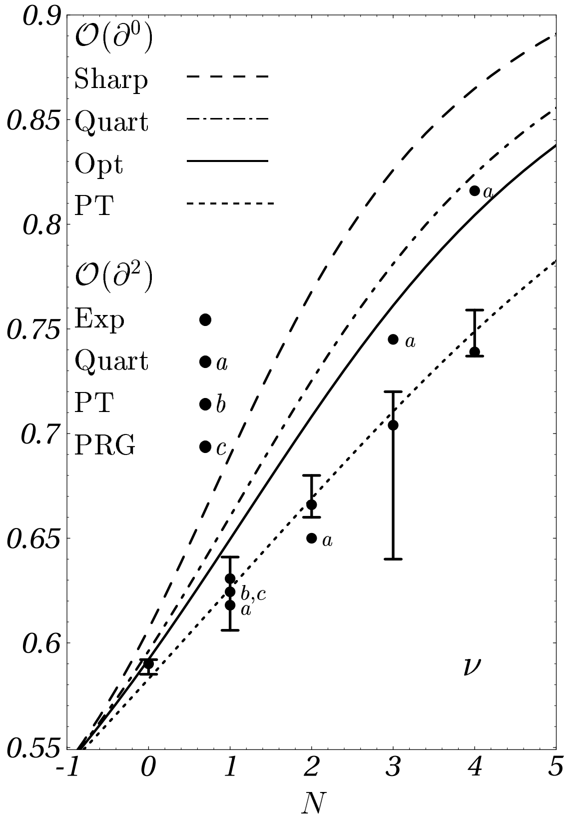


**Figure 4:** The critical exponent $\nu$; comparison of ERG results at $\mathcal{O}(\partial^0)$ (Sharp, Quartic, Opt) with a specific proper time RG (PT) [29], with ERG results to order $\mathcal{O}(\partial^2)$ (Quartic, Exp) [19, 27, 21], with results from the Polchinski RG (PRG) [10, 28] and a proper time RG (PT) to order $\mathcal{O}(\partial^2)$ [30], and with results from other methods (MC simulations, high temperature expansions, other field theoretical methods, experiments) [32].

quartic regulator converges less well for intermediate $N > 1$ (see also the related comments in ref. [27]). Supposedly, this problematic behaviour is due to the weak UV behaviour of the quartic regulator in $3d$. In the UV limit, $\partial_t R_{\rm quart}$ in (1.1) vanishes only polynomially, but not exponentially. This corresponds to an insufficiency in the integrating out of UV modes for (3.6), and may spoil the convergence. The optimisation ideas discussed in section 4 are essentially sensitive to the IR behaviour of the regulator, and cannot detect insufficiencies coming from the UV behaviour.

Finally, we turn to a proper time (PT) renormalisation group [34] for a specific operator cutoff discussed in ref. [29]. In contrast to the ERG flow (1.1), a PT flow is *not* an exact one (see ref. [29] for a comparison of ERG and PT flows). To leading order in the derivative expansion, only a subset of PT flows can be mapped onto ERG flows. Here, we consider a specific PT flow which cannot be mapped on a corresponding ERG flow, namely

$$\partial_t \Gamma_k = {\rm Tr}\ \exp -\Gamma_k^{(2)}/k^2 \,. \tag{7.1}$$

JHEP11(2001)059

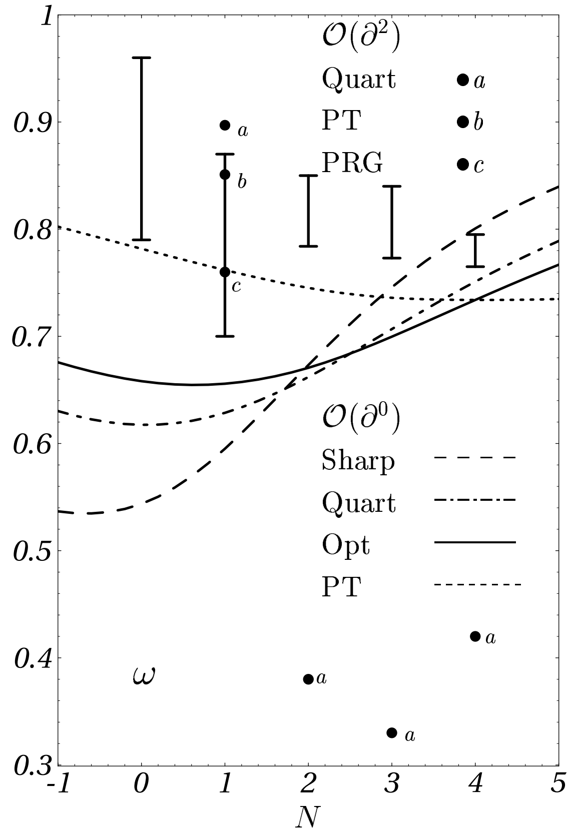


**Figure 5:** The smallest subleading eigenvalue $\omega$; comparison of ERG results to order $\mathcal{O}(\partial^0)$ (Sharp, Quartic, Opt) with a specific proper time RG (PT) to order $\mathcal{O}(\partial^0)$ [29], and predictions by other methods [32]. To order $\mathcal{O}(\partial^2)$, the ERG results (quartic regulator) are from [19], the Polchinski RG (PRG) result is from [10, 28], and the PT result from [30].

Currently, it is not known what approximation to an exact flow it describes [29]. To leading order in the derivative expansion, (7.1) turns into (3.11) with $\ell_{\rm PT}(\omega) = \Gamma(\frac{d}{2})\exp(-\omega)$ [29, 30]. Next, we apply the reasoning of section 4 to $\ell_{\rm PT}(\omega)$. Expanding the function $\ell_{\rm PT}(\omega)$ as in (4.1), the effective radius of convergence (4.3) of the flow (7.1) is trivially found to be $C_{\rm eff} = \infty$. Notice that $C_{\rm eff}$ is read off from $\ell_{\rm PT}(\omega)$, and not from a regularised propagator as in (4.4). The reason is simple. There exists no map which brings (7.1) into the form (1.1), not even to leading order in the derivative expansion [29]. The structure of the exact RG — with normalised regulators as defined in (3.4) and $C$ given by (4.4) — implies $C_{\rm ERG} < \infty$. Based on $C_{\rm eff} > C_{\rm ERG}$, we expect that the derivative expansion for (7.1) converges rapidly.

Critical exponents have been computed from (7.1) in refs. [29, 30] (see also ref. [31]). To order $\mathcal{O}(\partial^0)$, the PT results for $\nu$ ($\omega$) are displayed in figure 4 (figure 5) by a short dashed line. The PT results for $\nu(N)$ agree within a percent with those from other field theoretical methods [33]. The $N$ dependence of $\omega(N)$ is reproduced within 5%. Notice that the $\mathcal{O}(\partial^0)$ PT results nearly coincide with the $\mathcal{O}(\partial^2)$ ERG



JHEP11(2001)059

result for the exponential regulator. Furthermore, the leading order PT result for $N = 1$ agrees with the second order result of the Polchinski RG [10, 28]. To order $\mathcal{O}(\partial^2)$, for $N = 1$, the PT flow induces a negligible correction to $\nu$, and a 10% increase to $\omega$ [30]. While these results confirm our reasoning based on the arguments explained in sections 4 and 5, the numerical agreement with results obtained by other methods, or experiment, remains unclear. Currently, it is not understood why the derivative expansion of (7.1) should converge towards the *physical* scaling solution, bearing in mind that (7.1) is not an exact flow [29]. An answer to this question, however, is outside the range of the present study.

## 8. Discussion

We have studied the convergence of the derivative expansion. An adequate choice of the IR regulator turned out to be most vital for a good convergence. This is linked to the spurious regulator dependence of approximated flows, a dependence which would vanish for the integrated full flow. Indeed, changing the IR regulator within an approximated flow can be seen as a slight reorganisation of the derivative expansion. At a given order in the expansion, an optimised regulator effectively leads to the incorporation of contributions, which for other regulators would have appeared only to higher order. This is why certain regulators lead to better results already at lower orders. This interplay is used to *improve* the convergence of the derivative expansion.

We have applied these observations to the exact renormalisation group, to leading order in the derivative expansion and for subleading corrections proportional to the anomalous dimension. The leading order results for critical exponents of 3d O($N$) symmetric scalar theories are improved significantly for specific optimised regulators (figure 3). Furthermore, the back coupling of the anomalous dimension is reduced by a factor of 1/4 for generic optimised regulators, and by $1/(d+2)$ for the specific regulator (3.5). This provides an additional explanation for why higher order corrections remain small. For these reasons — in combination with the exactness of the flow — we expect that the corresponding critical exponents are closer to those of the physical theory.

This picture has explicitly been confirmed by establishing a link between the radius of convergence (a.k.a. the "gap") and the proximity of the corresponding value for $\nu_{\text{ERG}}$ to the physical value $\nu_{\text{phys}}$. The line of reasoning is entirely based on the structure of the flow (1.1), which contains all the universal information relevant at a scaling solution. In principle, our analysis applies as well for other physical observables which can be computed for small field amplitudes. As a side result, we found that the subleading critical exponents for $N = -2$ are non-universal, in contrast to the case $N = \infty$.



JHEP11(2001)059

For the proper-time renormalisation group, a reasoning analogous to the one in section 4 has been applied. We have considered the radius of convergence for amplitude expansions of the specific PT flow (7.1), which is found to be larger than for exact RG flows. This qualitative difference between exact and PT flows is linked to their fundamental inequivalence, even to this order of the approximation [29]. Still, the large effective radius of convergence explains why the derivative expansion for the PT flow (7.1) should have even better convergence properties. While this explains the convergence behaviour found in ref. [30] for $N = 1$, it remains to be clarified whether it converges towards the physical theory.

It would be interesting to apply these considerations to other theories, like scalar QED [35, 36, 37, 12], or quantum Einstein gravity [38], where recent investigations showed strong evidence for the existence of a non-trivial UV fixed point even in $d = 4$ dimensions. In contrast to the scalar theories studied here, the stability matrix at criticality discussed in [39] has complex eigenvalues for various classes of regulators, and a minimum sensitivity condition seems not to be applicable. Within the approximations employed in [39], we expect that the optimised flows of [15] should lead to a good prediction of the fixed point.

## Acknowledgments

This work has been supported by a Marie-Curie fellowship under EC contract no. HPMF-CT-1999-00404.

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JHEP11(2001)059